as a function on $X \times I$, is continuous. Because the determinant function is also continuous, that same $k \times k$ submatrix must be nonsingular for all points (x, t) in a neighborhood of $(x_0, 0)$ as claimed.

The proof of class (c) is virtually identical. For (d), recall that transversality with respect to Z may be locally translated into a submersion condition, so the proof of (d) is quite similar.

For (e), we need now only show that if f_0 is one-to-one, so is f_t if t is small enough. This proof is a relative of Exercise 10, Section 3. Define a smooth map $G: X \times I \longrightarrow Y \times I$ by $G(x, t) = (f_t(x), t)$. Then if (e) is false, there is a sequence $t_i \longrightarrow 0$ and distinct points $x_i, y_i \in X$ such that $G(x_i, t_i) = G(y_i, t_i)$. As X is compact, we may pass to a subsequence to obtain convergence $x_i \longrightarrow x_0$, $y_i \longrightarrow y_0$. Then

$$G(x_0, 0) = \lim G(x_i, t_i) = \lim G(y_i, t_i) = G(y_0, 0).$$

But $G(x_0, 0) = f_0(x_0)$ and $G(y_0, 0) = f_0(y_0)$, so if f_0 is injective, x_0 must equal y_0. Now, locally, we may work in Euclidean space. The matrix of $dG_{(x_0,0)}$ is just

$$\left(\begin{array}{c|c} & a_1 \\ d(f_0)_{x_0} & \vdots \\ & a_l \\ \hline 0 \ldots 0 & 1 \end{array}\right),$$

where the numbers a_j are not of interest. Since $d(f_0)_{x_0}$ is injective, its matrix must have k independent rows. Thus the matrix of $dG_{(x_0,0)}$ has $k+1$ independent rows, so $dG_{(x_0,0)}$ must be an injective linear map. Consequently, G is an immersion around $(x_0, 0)$ and thus must be one-to-one on some neighborhood of $(x_0, 0)$. But for large i, both (x_i, t_i) and (y_i, t_i) belong to this neighborhood, a contradiction.

Finally, we leave (f) for Exercise 8. Q.E.D.

EXERCISES

*1. Suppose that $f_0, f_1: X \longrightarrow Y$ are homotopic. Show that there exists a homotopy $\tilde{F}: X \times I \longrightarrow Y$ such that $\tilde{F}(x, t) = f_0(x)$ for all $t \in [0, \frac{1}{4}]$ and $\tilde{F}(x, t) = f_1(x)$ for all $t \in [\frac{3}{4}, 1]$. [HINT: Find a smooth function $\rho: \mathbf{R} \longrightarrow \mathbf{R}$ such that $\rho(t) = 0$ if $t \leq \frac{1}{4}$, $\rho(t) = 1$ if $t \geq \frac{3}{4}$. Now let F be any homotopy and set $\tilde{F}(x, t) = F(x, \rho(t))$.]

*2. Prove that homotopy is an equivalence relation: if $f \sim g$ and $g \sim h$, then $f \sim h$. [HINT: To join the homotopies together, you need Exercise 1. Why?]

*3. Show that every connected manifold X is *arcwise connected:* given any two points $x_0, x_1 \in X$, there exists a smooth curve $f: I \longrightarrow X$ with $f(0) = x_0$, $f(1) = x_1$. [HINT: Use Exercise 2 to show that the relation "x_0 and x_1 can be joined by a smooth curve" is an equivalence relation on X, and check that the equivalence classes are open.]

4. A manifold X is *contractible* if its identity map is homotopic to some constant map $X \longrightarrow \{x\}$, x being a point of X. Check that if X is contractible, then all maps of an arbitrary manifold Y into X are homotopic. (And conversely.)

5. Show that $\mathbf{R}^k$ is contractible.

6. A manifold X is *simply connected* if it is connected and if every map of the circle S^1 into X is homotopic to a constant. Check that all contractible spaces are simply connected, but convince yourself that the converse is false. (Soon we shall develop tools that easily prove the converse false.)

*7. Show that the *antipodal map* $x \longrightarrow -x$ of $S^k \longrightarrow S^k$ is homotopic to the identity if k is odd. HINT: Start off with $k = 1$ by using the linear maps defined by

$$\begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}.$$

8. Prove that diffeomorphisms constitute a stable class of mappings of compact manifolds; that is, prove part (f) of the Stability Theorem. [HINT: Reduce to the connected case. Then use the fact that local diffeomorphisms map open sets into open sets, plus part (e) of the theorem.]

9. Prove that the Stability Theorem is false on noncompact domains. Here's one counterexample, but find others yourself to understand what goes wrong. Let $\rho: \mathbf{R} \longrightarrow \mathbf{R}$ be a function with $\rho(s) = 1$ if $|s| < 1$, $\rho(s) = 0$ if $|s| > 2$. Define $f_t: \mathbf{R} \longrightarrow \mathbf{R}$ by $f_t(x) = x\rho(tx)$. Verify that this is a counterexample to all six parts of the theorem. [For part (d), use $Z = \{0\}$.]

*10. A *deformation* of a submanifold Z in Y is a smooth homotopy $i_t: Z \longrightarrow Y$ where i_0 is the inclusion map $Z \longrightarrow Y$ and each i_t is an embedding. Thus $Z_t = i_t(Z)$ is a smoothly varying submanifold of Y with $Z_0 = Z$. Show that if Z is compact, then any homotopy i_t of its inclusion map is a deformation for small t. Give a counterexample in the noncompact case (other than the triviality where $\dim Z = \dim Y$).

11. We shall have use for a direct generalization of the notion of homotopy. Suppose that $f_s: X \longrightarrow Y$ is a family of smooth maps, indexed by a parameter s that varies over a subset S in some Euclidean space. We say that $\{f_s\}$ is a *smooth family* of mappings if the map $F: X \times S \longrightarrow Y$, defined by $F(x, s) = f_s(x)$, is smooth. Check that the Stability Theorem generalizes immediately to the following: if f_0 belongs to any of the classes listed, then there exists an $\epsilon > 0$ such that f_s belongs to the same class if $|s_0 - s| < \epsilon$.

§7 Sard's Theorem and Morse Functions

The preimage of a regular value of the smooth map $f: X \longrightarrow Y$ is a nice submanifold of X. This simple fact has led us to a generalization of the notion of regularity—namely, transversality—which we hope will be a key to decipher some of the secrets of the topology of manifolds. But the regularity condition on values of f is a strong one. Perhaps the condition is so strong that regular values occur too rarely for our Preimage Theorem to be of much use. In fact, precisely the opposite is true, as guaranteed by the second deep theorem to be borrowed from advanced calculus.

Sard's Theorem. If $f: X \longrightarrow Y$ is any smooth map of manifolds, then almost every point in Y is a regular value of f.

The statement may sound vague but that will be rectified. First, we declare an arbitrary set A in $\mathbf{R}^l$ to have *measure zero* if it can be covered by a countable number of rectangular solids with arbitrary small total volume. Of course, a rectangular solid in $\mathbf{R}^l$ is just a cartesian product of l intervals in $\mathbf{R}^1$, and its volume is the product of the lengths of the l intervals. Thus A has measure zero if, for every $\epsilon > 0$, there exists a countable collection $\{S_1, S_2, \ldots\}$ of rectangular solids in $\mathbf{R}^l$, such that A is contained in the union of the S_i and

$$\sum_{i=1}^{\infty} \operatorname{vol}(S_i) < \epsilon.$$

The concept of measure zero is extended to manifolds via local parametrizations. An arbitrary subset $C \subset Y$ has *measure zero* if, for every local parametrization ψ of Y, the preimage $\psi^{-1}(C)$ has measure zero in Euclidean space. The condition really need not be verified for every parametrization, for it is not difficult to show that if $A \subset \mathbf{R}^l$ has measure zero and $g: \mathbf{R}^l \longrightarrow \mathbf{R}^l$ is a smooth map, then $g(A)$ has measure zero. (Proof in Appendix A.) It follows that C has measure zero, provided that it can be covered by the images of some collection of local parametrizations ψ_α satsifying the condition that $\psi_\alpha^{-1}(C)$ has measure zero for each α.